Equimatchable Graphs are C_{2k+1}-free for k≥4

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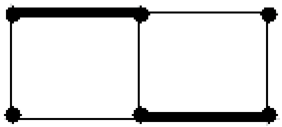
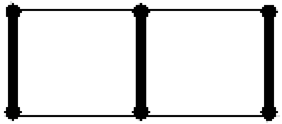
  - Equimatchable Graphs
  - Literature and our contribution
- Preliminaries
  - Gallai-Edmonds Decomposition
  - Related Structural Results
- Our Main Theorem

Introduction

- A ***matching*** M in G is a set of edges such that no two edges share a common vertex.
- A ***maximal matching*** is a matching M with the property that if any other edge is added to M, it is no longer a matching.



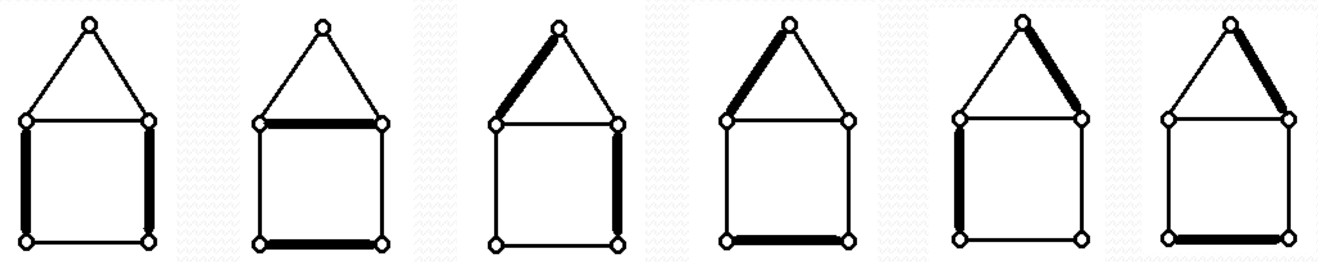
A maximal matching of size 2



A maximum matching of size 3

Equimatchable Graphs

A graph is ***equimatchable*** if all of its <u>maximal matchings</u> have the same size.



All maximal matchings have size 2

<u>Literature:</u>

- Recognition
- Characterization of equimatchable graphs with additional properties (connectivity, girth, etc.)

Our contribution

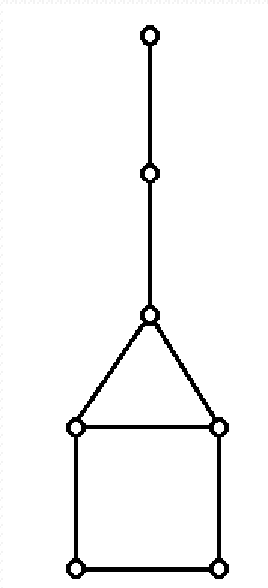
- ✓ The first family of **forbidden** induced subgraphs of equimatchable graphs (to the best of our knowledge).

- ✓ We show that equimatchable graphs do not contain **odd cycles of length at least nine**.

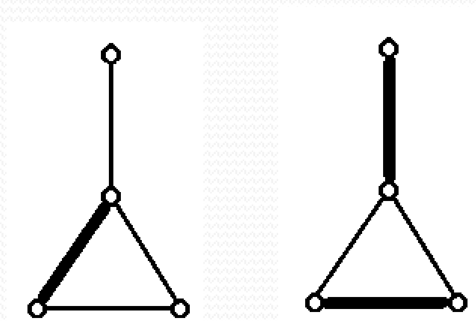
- ✓ The proof is based on
  - Gallai-Edmonds decomposition of equimatchable graphs (Lesk, Plummer, Pulleyblank, 1984)
  - The structure of factor-critical equimatchable graphs (Eiben, Kotrbcik, 2013)

Hereditary?

Being equimatchable is **not** a hereditary property, that is, it is not necessarily preserved by induced subgraphs.



Equimatchable



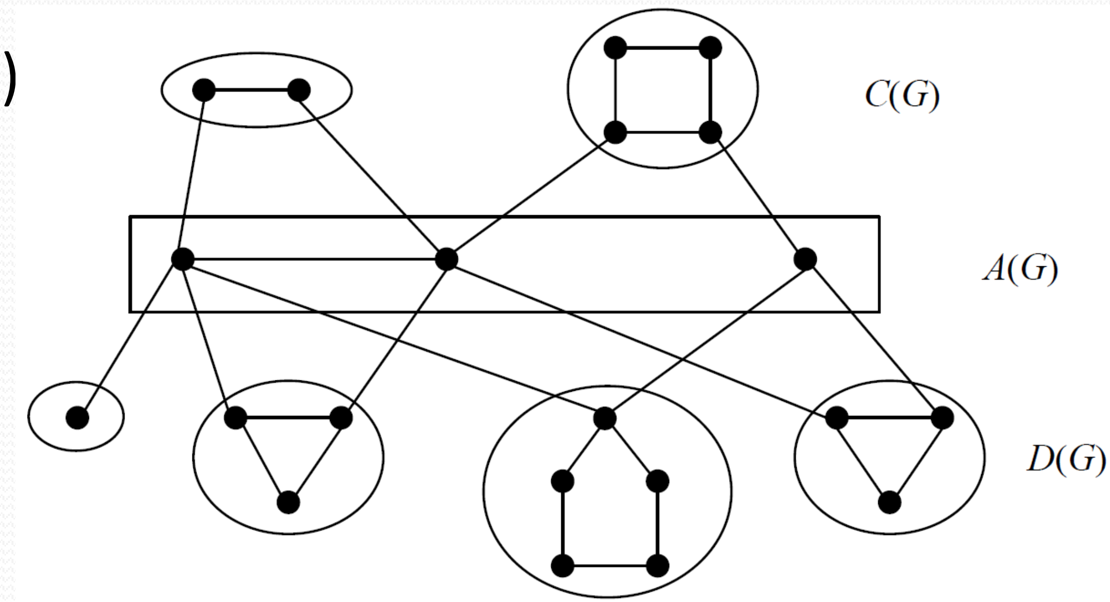
Not Equimatchable

Hereditary?

- It can be the case that there is no forbidden subgraph for being equimatchable at all.
- Finding a minimal non-equimatchable graph is not enough to say that it is forbidden for equimatchable graphs.
- We should find graphs that are not only non-equimatchable, but also not an induced subgraph of an equimatchable graph.

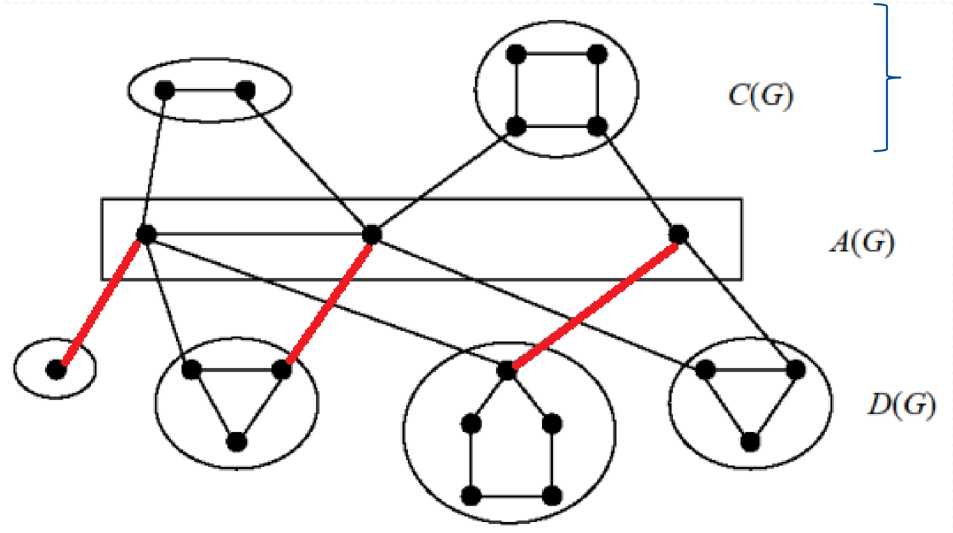
Gallai-Edmonds Decomposition

D(G) = the set of vertices of G that are **not** saturated by at least one **maximum** matching
A(G) = the set of vertices of V(G) \ D(G) with at least one neighbor in D(G)
C(G) = V(G) \ (D(G) ∪ A(G))



Gallai-Edmonds Decomposition

Theorem: (Lovasz, Plummer, 1986)
i) The connected components of D(G) are factor-critical.
ii) C(G) has a perfect matching.
iii) Every maximum matching of G matches every vertex of A(G) to a vertex of a distinct component of D(G).

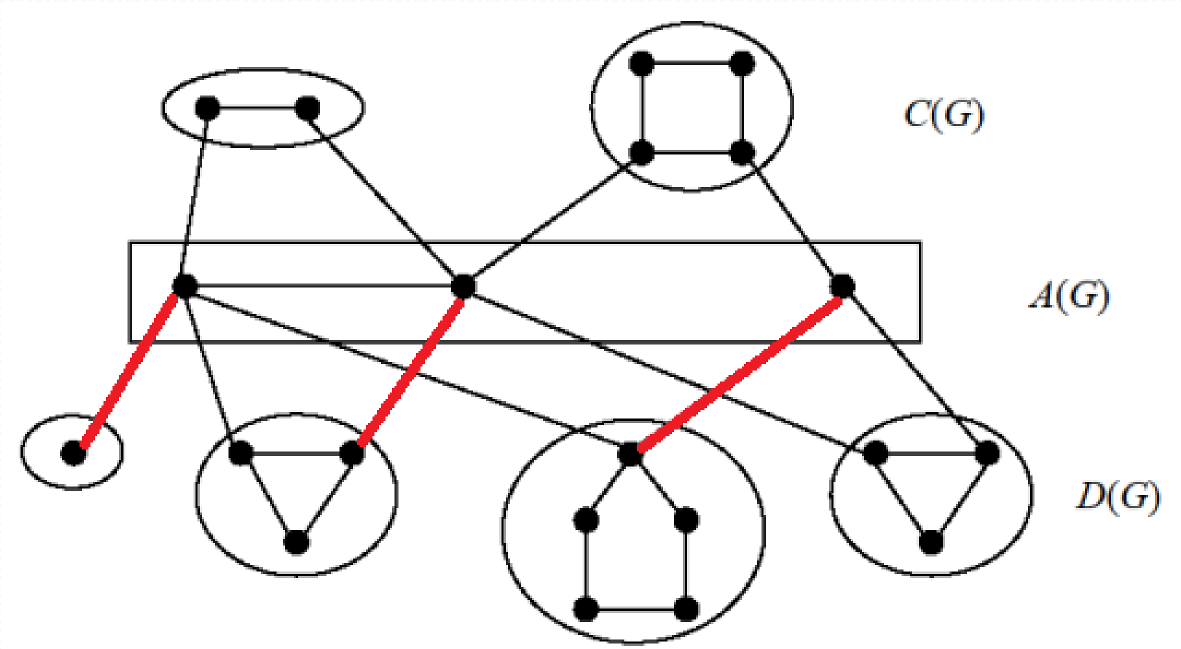


has a perfect matching

factor-critical

Preliminaries

Lemma: (Lesk, Plummer, Pulleyblank, 1984)
Let G be a connected equimatchable graph with no perfect matching. Then C(G) = Ø and A(G) is an independent set of G.



Preliminaries

Equimatchable graphs admitting a perfect matching = Randomly matchable (every maximal matching is perfect)

Lemma: (Sumner, 1979) A connected graph is randomly matchable if and only if it is isomorphic to a K_{2n} or a $K_{n,n}$ $(n \geq 1)$.

Preliminaries

- **Definition:** A graph G is ***factor-critical*** if G - u has a perfect matching for every vertex u of G.
- **Definition:** A matching M ***isolates*** v in G if v is an isolated vertex of G \ V(M). (M saturates N(v))
- **Lemma:** (Eiben, Kotrbcik, 2013) Let G be a connected, factor-critical, equimatchable graph and M be a matching isolating v. Then G \ (V (M) + v) is randomly matchable.

Road to the Main Result

Lemma: If G is an equimatchable graph with an induced subgraph C isomorphic to a cycle C_{2k+1} for some $k \geq 2$, then G is factor-critical.

Proof: Special structure of D(G) in the GED of equimatchable, non-factor-critical graphs (Lesk, Plummer, Pulleyblank, 1984)
→ at most 1 vertex of C_{2k+1} in every factor-critical component D_i
→ Vertices of C_{2k+1} alternate between a vertex in A and a vertex in D_i

Road to the Main Result

Given a factor-critical equimatchable graph, we need a special isolating matching with respect to a given subgraph (C_{2k+1}).

Lemma *Let v be a vertex of an equimatchable and factor-critical graph G, and let $C \subseteq V(G)$. There is a set of three vertex disjoint matchings M_1, M_2, M_3 and a partition of $N(v)$ into N_1, N_2, N_3 such that:*

i) $M_1 \cup M_2 \cup M_3$ *isolates* v,

ii) M_1 *is a perfect matching of* N_1,

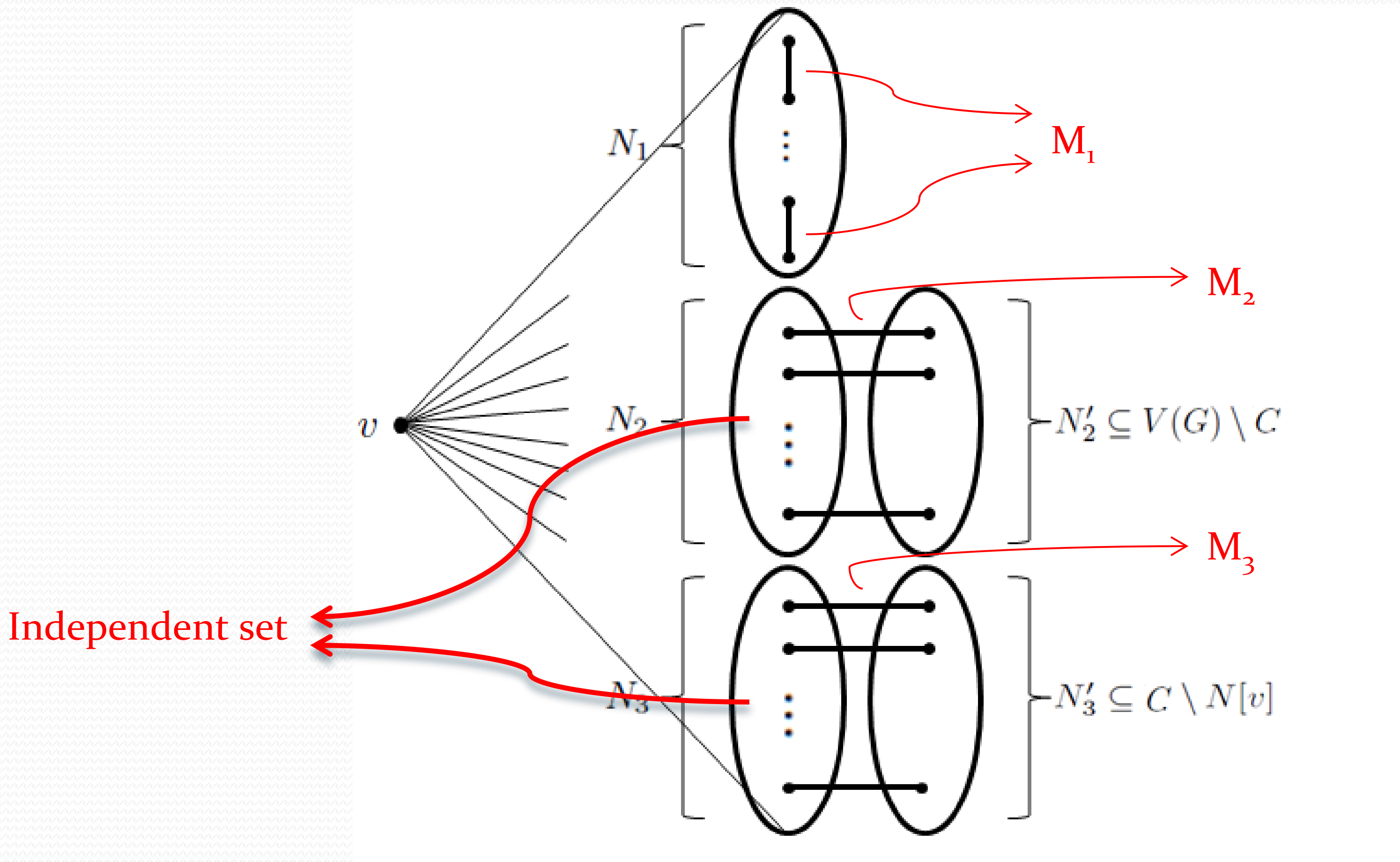
iii) M_2 *matches* N_2 *to some* N_2' *such that* $N_2' \cap C = \emptyset$,

iv) M_3 *matches* N_3 *to some* $N_3' \subseteq C \setminus N[v]$,

v) $N_2 \cup N_3$ *is an independent set, and*

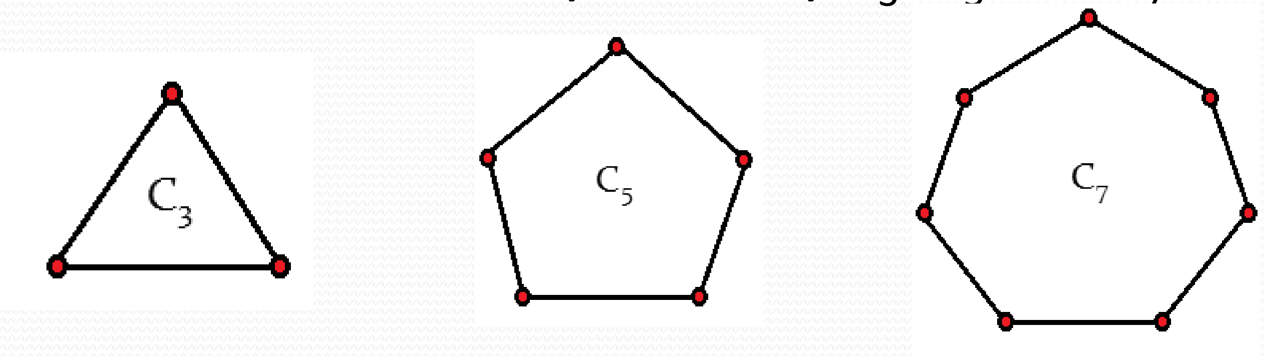
vi) $N(N_3) \subseteq N_1 \cup N_2' \cup C + v$.

Road to the Main Result



Road to the Main Result

It is easy to verify that C_{2k+1} is equimatchable if and only if $k \leq 3$.
In other words, for odd cycles, only C_3, C_5 and C_7 are equimatchable.



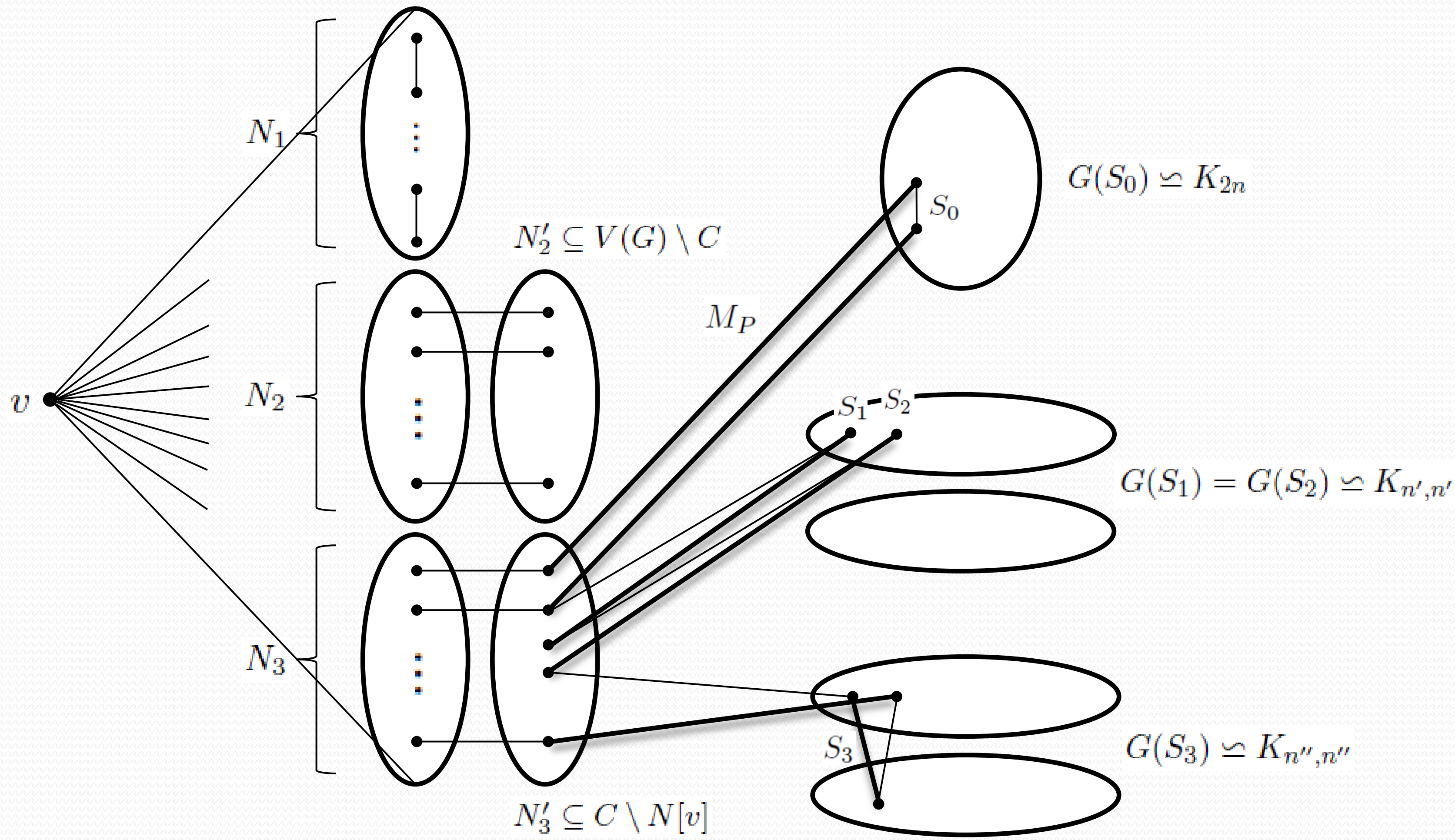
We prove a stronger result;

C_{2k+1} is not an induced subgraph of an equimatchable graph whenever $k \geq 4$.

Main Theorem

Theorem: Equimatchable graphs are C_{2k+1}-free for $k \geq 4$.

Proof: Let G be an equimatchable graph and let C be an induced odd cycle of G with at least 9 vertices.
→ Then, G is factor-critical. Therefore, every maximal matching of G leaves exactly one vertex exposed.
→ Construct matchings such that the removal of their endpoints disconnects G into at least two odd connected components.
→ This implies the existence of maximal matchings leaving at least two vertices of G exposed, leading to a contradiction.


N_1
N_2
N_3
v
$N_2' \subseteq V(G) \setminus C$
$N_3' \subseteq C \setminus N[v]$
M_P
S_0
$G(S_0) \simeq K_{2n}$
S_1 S_2
$G(S_1) = G(S_2) \simeq K_{n',n'}$
S_3
$G(S_3) \simeq K_{n'',n''}$

Main Theorem

Proof (contd.):

Let v be any vertex of the cycle C_{2k+1}

Let $P = C\backslash N[v]$ denote the path isomorphic to a P_{2k-2} obtained by the removal of v and its two neighbors from the cycle C.

Recall that $N_3' \subset C\backslash N[v] = P$

Denote by M_P the unique perfect matching of P

Main Theorem

Proof (contd.):

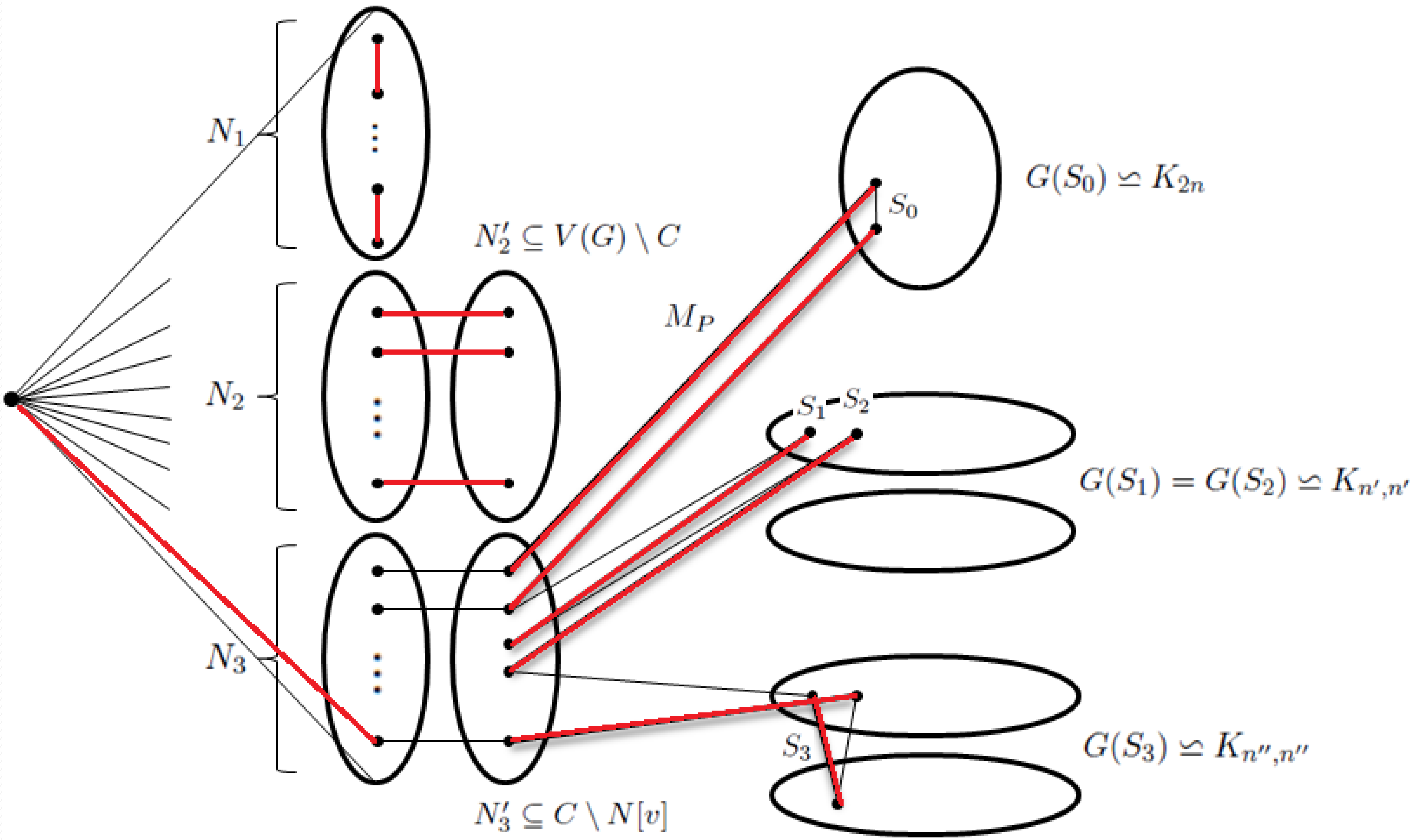
4 cases according to the number of vertices of N_3: $|N_3| \geq 3$, $|N_3| = 2$, $|N_3| = 1$, $|N_3| = 0$

Case $|N_3| \geq 3$:

Let $u \in N_3$ and consider the matching $M_1 \cup M_2 \cup M_P + uv$.

The removal of this matching leaves at least 2 isolated vertices (odd components) in $N_3 - u$.

CONTRADICTION !!



N_1
N_2
N_3
$N_2' \subseteq V(G) \setminus C$
$N_3' \subseteq C \setminus N[v]$
M_P
S_0
$G(S_0) \simeq K_{2n}$
S_1
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$G(S_1) = G(S_2) \simeq K_{n',n'}$
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$G(S_3) \simeq K_{n'',n''}$

Thank you for listening…